I. PRESENT VALUE

Introduction

In many applications, a policy decision requires the analyst to evaluate elements of the problem that occur in different time periods. Often, some or all of these elements are readily convertible to dollar figures, but sometimes they are not. These are two of the thornier problems in benefit-cost analysis: how do we value inputs or outcomes when they are not easily converted into pecuniary (dollar) terms, and how do we place values that occur in different time periods on a comparable basis.

This note addresses the second question. It describes in very simple terms the technique known as *present value analysis*, and illustrates with an example drawn from environmental policy.

DISCOUNTING: CONVERTING FUTURE LUMP SUM PAYMENTS TO PRESENT VALUES

Present value analysis puts dollar-denominated data that occur in some period other than the present on comparable footing with data denominated in current period dollars. When the time periods are "discrete" we are doing "*discrete* compounding" or "discrete discounting." When they are "continuous" (i.e., infinitesimally small, instantaneous, etc.), we use *continuous* compounding or discounting.

The valuation technique is motivated with the following simple question:

> "How much money would you have to be paid today in order to be indifferent between that amount of money and a payment of $X one period hence?"

The immediate response to such a question might be

> "How much will I earn with that money if I receive it today, rather than wait for one period to elapse?"

In the example, if X was $1000, and the rate of return on your investment today was 10%, then you would observe most individuals preferring, for example, $950 today to $1000 in a year, and preferring $1000 in a year to, say, $800 today. Why? Obviously, the $950 received now would earn $95 over the course of the year, and the choice between $1000 in a year vs. $1045 in a year should be a pretty easy one. Similarly, if you receive $800 now and invest it at a 10% rate of return, you would have $880 a year from now, and that would hardly be preferable to $1000 at that time.

We can vary the question in an important way by asking

> "How much would you have to receive today to be indifferent between that sum and $1000 a year from now?"

If we call that amount V_0, the answer to the question solves the equation

$$V_0(1.10) = \$1000,$$

or

$$V_0 = \frac{1}{(1.10)}\$1000.$$

Here, V_0 is the present value of $1000 one year from today, given a 10% rate of return.

If the $1000 was going to be received two periods later, the present value would be computed using

$$V_0(1.10)(1.10) = V_0(1.10)^2 = \$1000.$$

Here, the returns are *compounded.* That is, you earn a 10% return on the original investment for two periods, plus a 10% return on the year 1 return too. The present value of $1000 two periods from now with a 10% rate of return is

$$V_0 = \frac{1}{(1.10)^2}\$1000.$$

In general, the present value of $1 (we use $1 and then just multiply the present value of the dollar by the actual dollar amount to get the present value of the actual dollars) T years from now at a rate of return equal to $r\%$ is given by

$$V_0 = \frac{\$1}{(1+r)^T}$$

Alternatively, we could write this as

$$V_0 = \$1(1+r)^{-T}.$$

The rate of return here is most commonly referred as the *discount rate*, and the process of calculating the present value of some amount of money to be received in the future is called *discounting.*

For any amount received at some point in the future, the present value with discrete discounting is

$$V_0 = V_T(1+r)^{-T}.$$

In continuous time, present values of lump sums are exponential function of the discount rate, r, and the time to maturity, T. That is, with continuous discounting, the present value of $1 at some time in the future is given by

$$V_0 = e^{-rT}.$$

For any amount of money paid in the future, the present value with continuous discounting is

$$V_0 = V_T e^{-rT}.$$

Exercise: What discount rate would render you indifferent between $1000 today, and $1250 three years from today? [Note: this particular discount rate has a name – the *internal rate of return.*]

COMPOUNDING: CONVERTING CURRENT LUMP SUMS TO FUTURE VALUES

At some point, nearly everyone starts to think about the following question:

"If I invest $X today and earn a rate of return of $r\%$, how much will I have in T years."

To answer this question, we invert the present value calculations, and reflect the fact that the returns will be compounded (i.e., you earn "interest on the interest") every year. The solution is given by the expression

$$V_T = V_0(1+r)(1+r)\ldots(1+r),$$

where there is one "compounding factor" $(1+r)$ for each of the T periods. This can be written as

$$V_T = V_0(1+r)^T.$$

If the returns are compounded continuously, the future value is given by the exponential expression

$$V_T = V_0 e^{rT}$$

Note the symmetry among the various formulas.

Consider the following, from the *Faster Times* (May 24, 2010, by Joshua Brown)

On May 24th 1626, **Peter Minuit** (also spelled 'Minuet') purchased the island of Manhattan for the equivalent of $24 worth of beads and trinkets. Even adjusted for inflation, this is probably the real *Greatest Trade Ever*, with apologies to **John Paulson**.

Here's some historical/legendary color on the trade:

On May 24 1626, he is credited with the purchase of the island from the natives - perhaps from a Metoac band of Lenape known as the "Canarsee" - in exchange for trade goods valued at 60 guilders. This figure is known from a letter by a member of the board of the Dutch West India Company Peter Stuyvesant to the States-General in 1626; in 1846 the figure was converted by a New York historian to $24, and "a variable-rate myth being a contradiction in terms, the purchase price remains forever frozen at twenty-four dollars," as Edwin Burrows and Mike Wallace remarked: a further embellishment in 1877 converted the figure into "beads, buttons and other trinkets." A contemporary purchase of rights in Staten Island, New York, to which Minuit was also party, involved duffel cloth, iron kettles and axe heads, hoes, wampum, drilling awls, "Jew's Harps," and "diverse other wares". "If similar trade goods were involved in the Manhattan arrangement," Burrows and Wallace surmise, "then the Dutch were engaged in high-end technology transfer, handing over equipment of enormous usefulness in tasks ranging from clearing land to drilling wampum." If the island was purchased from the Canarsees, they would have been living on Long Island and maybe passing through on a hunting trip. The "purchase" was understood differently by both parties, the local group having no conception of alienable real estate, as is always pointed out in modern accounts of the supposed transaction.

Here's the punchline...after a series of brutal conflicts between the new Dutch inhabitants of Manhattan and various tribes who really had no concept of real estate or land ownership ensued, Peter Minuit was recalled back to the Netherlands to explain himself. He was then fired. Sometimes a bargain isn't really a bargain.

Source:

Peter Minuit Entry (Free Dictionary)

Exercise: Compute the value today of $24 in 1626 (assume January 1), compounded at a rate of 6%/year. Compute it for a couple other rates, e.g., 3% and 12%. What happens as you double and halve the compounding rate? [Use continuous compounding.]

DISCOUNTING A FLOW OF PAYMENTS

Discrete Time

Sometimes costs and benefits accrue over several different periods in the future. The most common example is a recurring stream of payments made or received. If these payments are made regularly for T periods, they can be viewed as a sequence of lump sum payments, and the formulas above apply directly. The present value of a stream of payments, $v_1, v_2, v_3, \dots v_T$ is given by the expression

$$V_0 = v_1(1+r)^{-1} + v_2(1+r)^{-2} + v_3(1+r)^{-3} + \cdots v_T(1+r)^{-T}.$$

If the payments are identical in each period (sometimes called an "annuity"), equal to v_R, then we can write this expression as

$$V_0 = v_R[(1+r)^{-1} + (1+r)^{-2} + (1+r)^{-3} + \cdots (1+r)^{-T}$$

or, using "summation notation,

$$V_0 = v_R \sum_{t=1}^{T}(1+r)^{-t}.$$

Exercise (hard): Show that the previous expression can be written as

$$V_0 = v_R \left[\frac{1-(1+r)^{-T}}{r}\right]$$

Continuous Time

In continuous time, the present value of an annuity v_R is computed similarly. Here the summation is done using "integration"

$$V_0 = v_R \int_0^T e^{-rt} dt$$

This integral has the solution

$$V_0 = \frac{v_R}{r}[1 - e^{-rT}]$$

Note the symmetry again between the discrete and continuous time formulas for the present value of an annuity.

COMPOUNDING A FLOW OF PAYMENTS: FUTURE VALUES OF ANNUITIES

We can compute future values of flows of funds, in particular of annuities, in a similar manner as for lump sums. The formulas are

Discrete time: $$V_0 = v_R \left[\frac{(1+r)^T - 1}{r}\right]$$

and

Continuous time: $$V_0 = v_R \left[\frac{e^{rT} - 1}{r}\right]$$